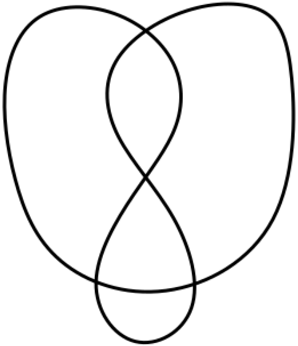
To Knot or Not to Knot

Ana Wright

26, February 2020

The Knotting/Unknotting Game

Given a knot projection, two players take turns resolving crossings one at a time. One player wins if the resulting knot diagram is the unknot. The other player wins if the resulting diagram is anything nontrivial.



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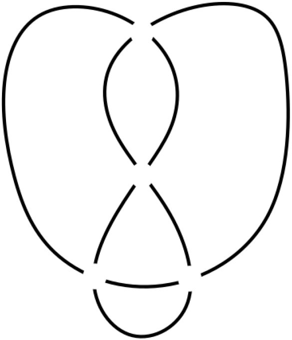

Examples

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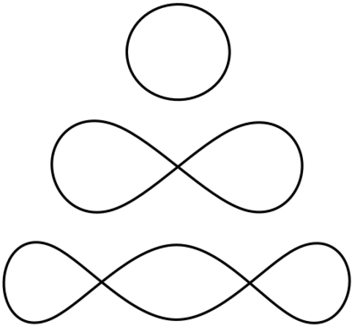

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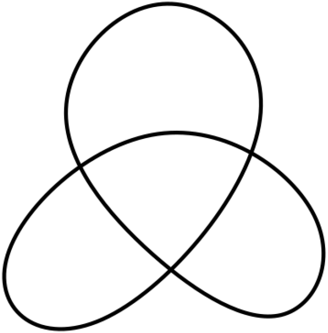

Examples

Give a knot projection where the player who goes first will always win. Give an example where the second player will always win.

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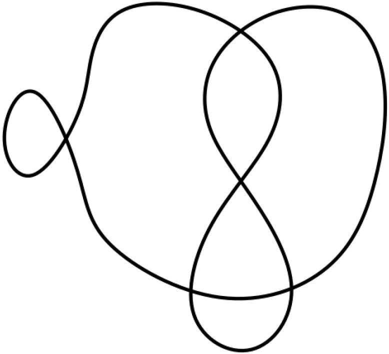
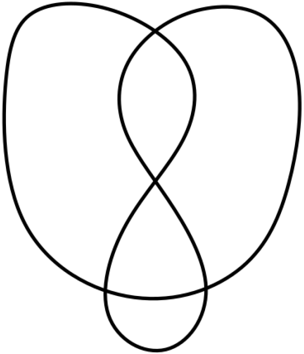


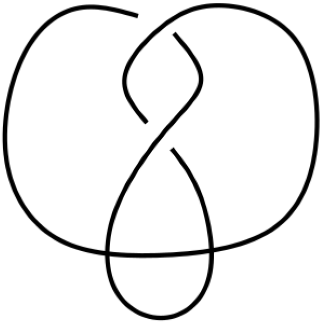


My Question (from my directed reading program)

Does there exist a knot projection such that the knotter will always win?

Pseudodiagrams

Def: A pseudodiagram is a knot projection where some subset of the crossings have crossing information.

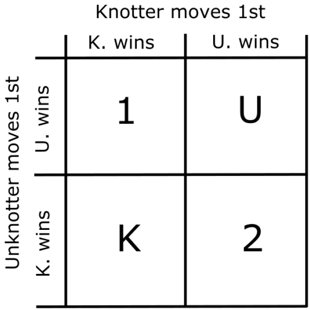


Outcome Classes

Each pseudodiagram can be placed in an outcome class. This is the outcome of perfect play on the remaining unresolved crossings.

Unknotter moves 1st \ Knotter moves 1st	K. wins	U. wins
U. wins	1	U
K. wins	K	2

Outcome Classes



Notice that the knotter can safely move to a pseudodiagram in $K2 := K \cup 2$ and wants to recieve a pseudodiagram in $K1 := K \cup 1$.
Similarly, the unknotter can safely move to a pseudodiagram in $U2 := U \cup 2$ and wants to recieve a pseudodiagram in $U1 := U \cup 1$.

Moves

The following moves do not affect the strategy of the game, so we will consider pseudodiagrams equivalent under these moves and the non-pseudo Reidemeister moves:

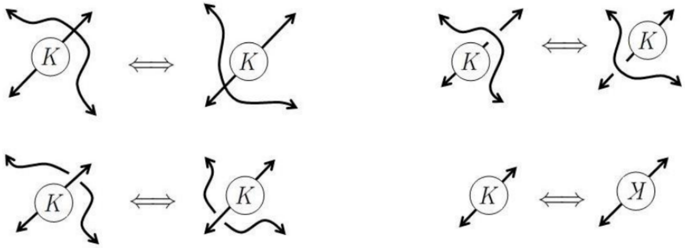
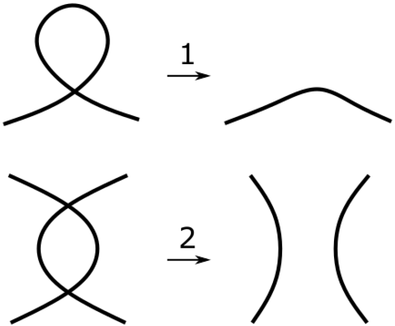


Image credit: Will Johnson

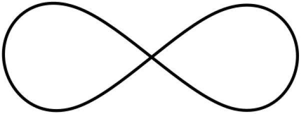
Moves

These pseudo Reidemeister moves do affect the strategy of the game:

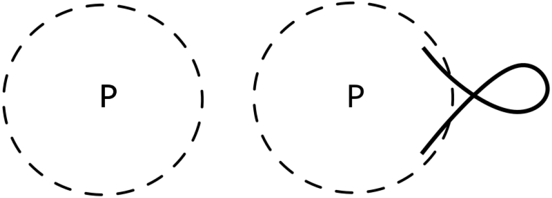


The Effect of Pseudo Reidemeister Moves

Let $\star$ represent the pseudodiagram below.

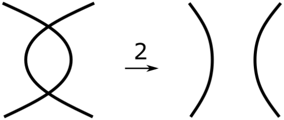


Lemma (Johnson, 2018):
If a pseudodiagram P is in $U2$, then $P\#\star$ is in $U1$.
If a pseudodiagram P is in $K2$, then $P\#\star$ is in $K1$.



The Effect of Pseudo Reidemeister Moves

Suppose $P \xrightarrow{2} Q$



Lemma (Johnson, 2018):
If P and Q have an even number of unresolved crossings, then

$$Q \in U2 \Rightarrow P \in U2$$
$$Q \in K2 \Rightarrow P \in K2.$$

If P and Q have an odd number of unresolved crossings, then

$$Q \in U1 \Rightarrow P \in U1$$
$$Q \in K1 \Rightarrow P \in K1.$$

The Effect of Pseudo Reidemeister Moves

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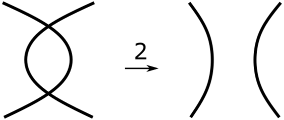
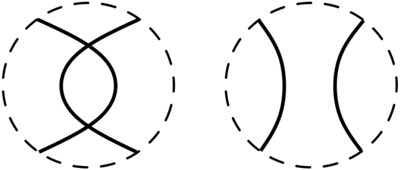


Restated Lemma (Johnson, 2018):
P is no worse than *Q* for the player who will make the last move of the game.

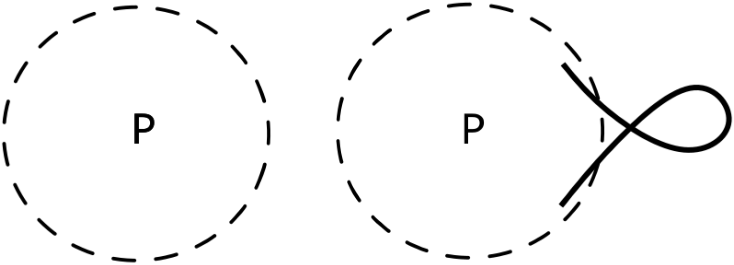


Definitions

Given a pseudodiagram P, define P^1 and P^0 such that

$$\{P, P\#\star\} = \{P^0, P^1\}$$

where P^0 has an even number of unresolved crossings and P^1 has an odd number of unresolved crossings.

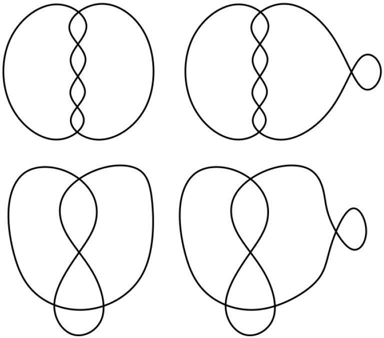


Definitions

Given a pseudodiagram P, define $o(P) \in \{1, 2, K, U\}$ to be the outcome of P.

Define $(o(P), o(P\#\star))$ to be the **extended outcome.**

Define $(o(P^0), o(P^1))$ to be the **normalized outcome.**



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Lemma (Johnson, 2018):
Pseudo Reidemeister I moves have no effect on normalized outcomes.

$$(P\#\star)^0 = \begin{cases} P^0 & \text{if } P \text{ has odd unresolved crossings} \\ P^0\#\star\#\star & \text{if } P \text{ has even unresolved crossings} \end{cases}$$

$$(P\#\star)^1 = \begin{cases} P^1 & \text{if } P \text{ has even unresolved crossings} \\ P^1\#\star\#\star & \text{if } P \text{ has odd unresolved crossings} \end{cases}$$

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Definitions

For any pseudodiagram P, define $X(P), Y(P) \in \{1, 2, 3\}$ such that we have the table of normalized outcomes below.

Y \ X	1	2	3
1	(U,U)	(1,U)	(1,2)
2	(U,1)	(1,1)	(1,K)
3	(2,1)	(K,1)	(K,K)

Larger values of $X(P)$ and $Y(P)$ are better for the knotter and smaller values of $X(P)$ and $Y(P)$ are better for the unknotter.

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Results

Theorem (Johnson, 2018):
If $P \xrightarrow{1} Q$ then

$$X(P) = X(Q)$$
$$Y(P) = Y(Q)$$

and if $P \xrightarrow{2} Q$ then

$$X(P) \leq X(Q)$$
$$Y(P) \geq Y(Q)$$

so if P reduces to Q via pseudo Reidemeister moves, then

$$X(P) \leq X(Q)$$
$$Y(P) \geq Y(Q)$$

Proof Sketch

- Suppose $P \xrightarrow{2} Q$. Then $P^0 \xrightarrow{2} Q^0$ and $P^1 \xrightarrow{2} Q^1$.
- By our lemma on the effect of pseudo Reidemeister II moves, we have

$$
\begin{aligned}
Q^0 \in U2 &\Rightarrow P^0 \in U2 \\
Q^0 \in K2 &\Rightarrow P^0 \in K2 \\
Q^1 \in U1 &\Rightarrow P^1 \in U1 \\
Q^1 \in K1 &\Rightarrow P^1 \in K1
\end{aligned}
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- Using our table from earlier, this means

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X(Q) = 1 &\Rightarrow X(P) = 1 \\
X(Q) < 3 &\Rightarrow X(P) < 3 \\
Y(Q) = 3 &\Rightarrow Y(P) = 3 \\
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Why My Examples Haven't Been Working

Corollary (Johnson, 2018):
Let P be a pseudodiagram that reduces to the unknot via pseudo Reidemeister moves. If P has even unresolved crossings, then its outcome is either U or 2. If P has odd unresolved crossings, then its outcome is either U or 1.

In particular, P does not have outcome K.

Proof:

- The unknot has normalized outcome (U, U), so it has X and Y values of 1.
- By the main theorem, $X(P) \leq 1$ and $Y(P) \geq 1$, so $X(P) = 1$.
- P^0 is in $U2$ and P^1 is in $U1$

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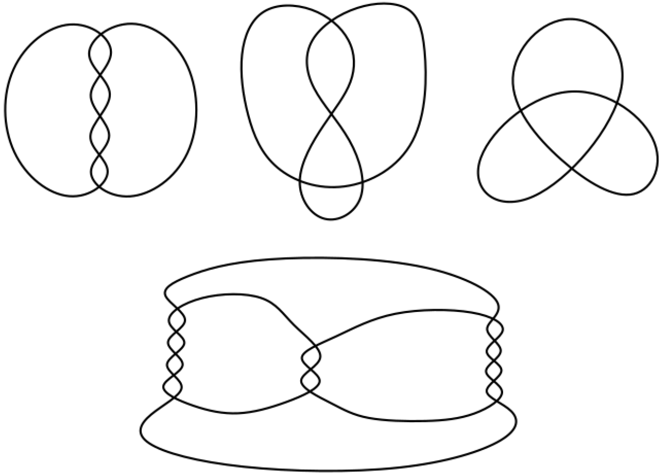

I Have My Answer

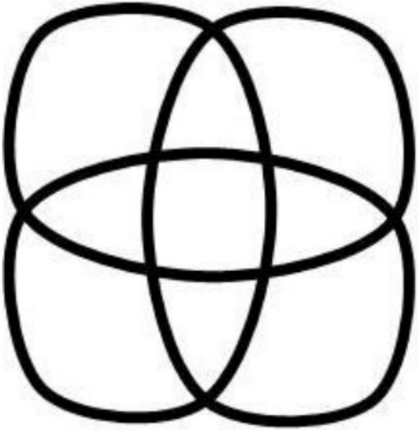

Image credit: Will Johnson

References

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(2018)

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The Knotting-Unknotting Game played on Sums of Rational Shadows
(2018)